Master Coaching

MASTER COACHING

Master Pythagoras

Pythagoras' Theorem $\quad A^2 + B^2 = C^2$

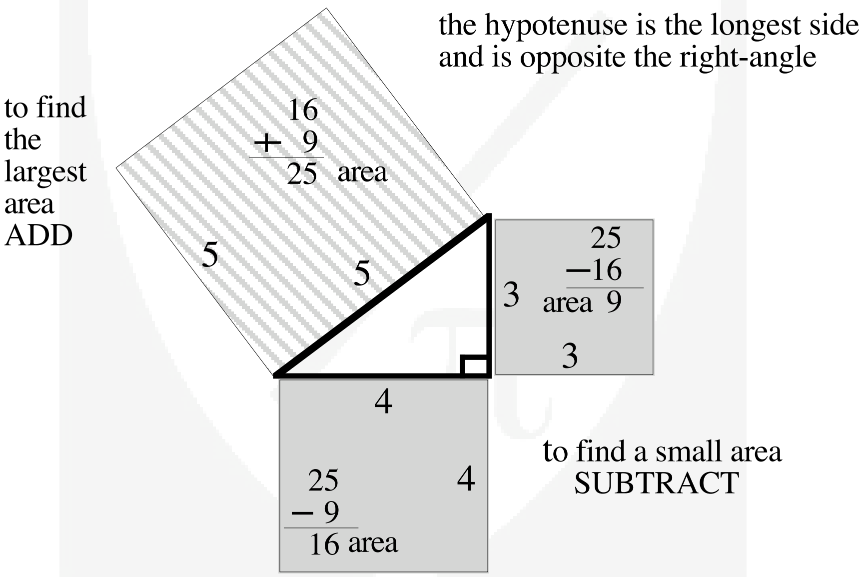
Most students learn $a^2 + b^2 = c^2$ without understanding the geometrical concepts involved.
It is *easier* to understand this topic from a geometrical sense without resorting to algebra.

We are talking in fact about three <u>areas</u> when we state :

> *"The square on the hypotenuse
> equals the sum of the squares
> on the other two sides."*

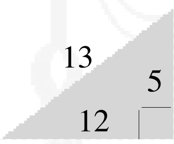
Note that this theorem needs triangles that have a **right angle** ... there must be a $90°$ angle!

Draw a diagram and actually write down the areas.
The two smaller squares added together equal the size of the big one
Your eyes will help explain the concept of Pythagoras' theorem.



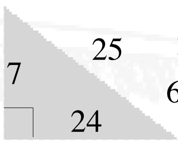
If all three sides are *whole* numbers they are said to be a **Pythagorean triad**.

eg



$13^2 = 5^2 + 12^2$

$169 = 25 + 144$



$25^2 = 24^2 + 7^2$

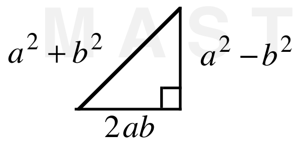
$625 = 576 + 49$

Some more of the infinite number of different Pythagorean triads are :
6 , 8 , 10   8 , 15 , 17   20 , 21 , 29   33 , 56 , 65

Pythagoras was the son of a Greek merchant. Born 570 BC, he died 501 BC.

He probably learned this triangle fact while in Egypt ... he was able to *prove* it was true.
He taught the'secret knowledge' at his school in a Greek colony at the foot of modern Italy.
The philosophy of Pythagoras is interesting and has influenced people for millenia.

A **formula for triads** is shown here ;



... just choose $a > b$

some examples are
in the table at right ...

a	b	a^2+b^2	a^2-b^2	$2ab$
2	1	5	3	4
3	2	13	5	12
4	1	17	15	8
4	3	25	7	24
5	2	29	21	20
5	4	41	9	40
7	6	85	13	84
9	2	85	77	36
47	16	2465	1953	1504

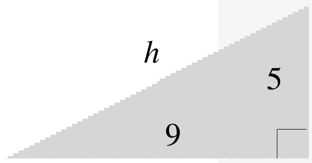
If your Algebra is good
you will see that ...

$$(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$$
$$a^4+2a^2b^2+b^4 = (a^4-2a^2b^2+b^4)+4a^2b^2$$

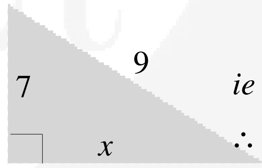
Questions often use a triad multiple, eg 60 , 80 ,100 $\quad 0\cdot5\ , 1\cdot2\ , 1\cdot3 \quad \frac{7}{11}\ , 2\frac{2}{11}\ , 2\frac{3}{11}$

A little known fact : *if* a ,b , c *are a triad then* $a \times b \times c$ *is divisible by 60 (ie by 3, by 4 and by 5)*

Usually we get *irrational* lengths for a side :



$h^2 = 5^2 + 9^2$
ie $h^2 = 106$
$\therefore \quad h = \sqrt{106}$



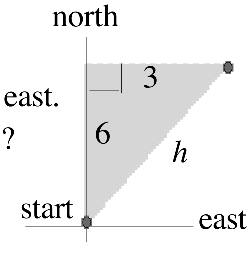
$x^2 = 9^2 - 7^2$
ie $x^2 = 32$
$\therefore \quad x = \sqrt{32}$

....

Enjoy these sample applications :

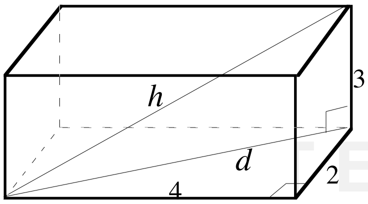
Wandering in the Bush :

Tina walked 6 km north and then 3km east.
How far is she from her starting point ?



$h^2 = 6^2 + 3^2$
ie $h^2 = 45$
$\therefore \quad h = \sqrt{45} \approx 6\cdot7$
ie about $6\cdot7$ km away

in 3D ...The Box : what is the longest thin pole that can be placed in a container that is 2m by 3m by 4m in the shape of a rectangular prism?



first we get $d^2 = 4^2 + 2^2$, then again $h^2 = d^2 + 3^2$

ie $h^2 = 4^2 + 2^2 + 3^2$ so $h = \sqrt{29}$

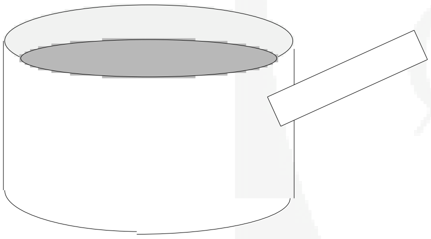
Check that it doesn't matter what order you apply the two doses of Pythagoras Theorem.

....

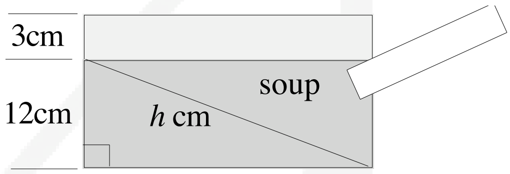
in the Kitchen : I am using a pot that is 15 cm deep and has a radius of 10 cm.
This cylindrical pot has 12 cm of hot soup in it.
What length should my spoon be?

First we draw a diagram ... this should be quick but accurate ...
... re-read the question ... with circles, always check whether it is a radius or a diameter ...

3d



cross-section , ie a side view



We can see a right-angled triangle that we can use to find the diagonal length h cm.
If the spoon is shorter than h cm then it can slip under the top surface of the soup.
A real spoon has a thickness, unlike our mathematical diagonal, but you get the idea.

Now we apply Pythagoras' Theorem :

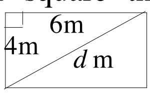
we write $h^2 = 20^2 + 12^2$	or $l^2 = 20^2 + 15^2$ to rim of pot
$h^2 = 400 + 144$	$l^2 = 400 + 225$
$h^2 = 544$	$l^2 = 625$
$h = \sqrt{544}$	$l = \sqrt{625}$
$h = 23 \cdot 32$ to 2 decimal places	$l = 25$ exactly

Perhaps a 27cm spoon, 2cm out of the pot, is a suitable length.

....

Shed building : The floor is a rectangle 6m by 4m. The diagonal is $7 \cdot 29$ m.
Is there a problem?

if 'square' then

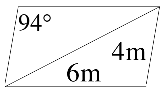


$d^2 = 6^2 + 4^2$

$d = \sqrt{52}$

$d = 7 \cdot 21$ to nearest cm

so the shed is not 'square'
ie the corners are not 90°

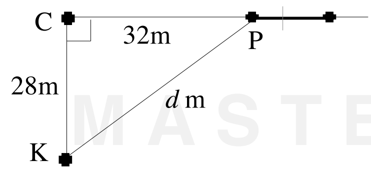


One diagonal was too long ...
the other would be too short

They must be the same length.

At the Footie :

A try has been scored next to the sideline in the corner.
The kicker chooses to attempt the conversion from a spot 28m back from the try line.
The field is 70m wide. The posts are 6m apart and the crossbar is 3m high.



$$d^2 = 28^2 + 32^2$$
$$d = \sqrt{1808}$$
$$d = 42 \cdot 52 \text{ ... 2dp}$$

It is $42 \cdot 52$m travelling along the ground to hit the foot of the near post!
The ball needs to get *over* the crossbar to score ponts.

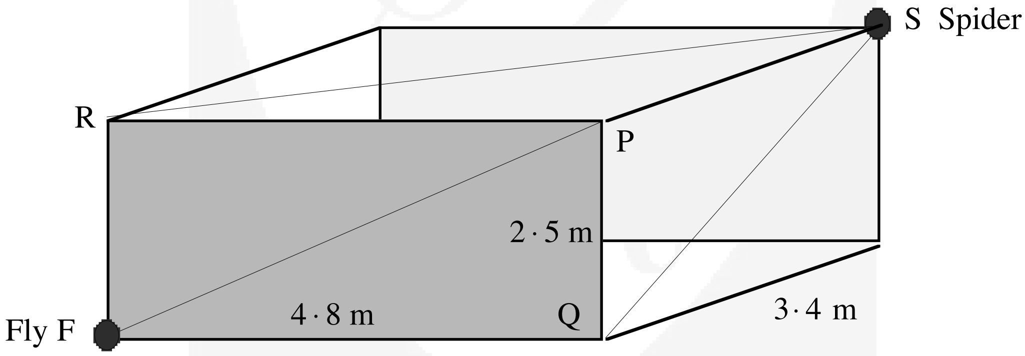
We used simple Pythagoras ; a more accurate answer needs the mathematics of 'Projectiles.
'Projectiles' involves *parabolas* and *gravity* and *trigonometry* and *calculus* and more.

....

Spider and Fly : ... a famous puzzle!

Spider is in a top corner of a room that is $3 \cdot 4$ m wide by $4 \cdot 8$ m long by $2 \cdot 5$ m high.
Fly is asleep on the floor in the corner furthest from Spider.

What is the shortest distance that Spider must walk to reach Fly.



Using Pythagoras $PF^2 = 2 \cdot 5^2 + 4 \cdot 8^2$, $SQ^2 = 2 \cdot 5^2 + 3 \cdot 4^2$ and $SR^2 = 4 \cdot 8^2 + 3 \cdot 4^2$
$PF = \sqrt{29 \cdot 29}$ $SQ = \sqrt{17 \cdot 81}$ $SR = \sqrt{34 \cdot 60}$

Thus $SPF = 3 \cdot 4 + \sqrt{29 \cdot 29} = 8 \cdot 81$, $SQF = \sqrt{17 \cdot 81} + 4 \cdot 8 = 8 \cdot 22$ and $SRF = \sqrt{34 \cdot 60} + 2 \cdot 5 = 8 \cdot 38$

However, the shortest journey will *not* be one of the three paths above.
You can draw various *nets* of the prism and draw a straight line SF to solve this problem,
but we wanted to demonstrate the usefulness of Pythagoras' Theorem.

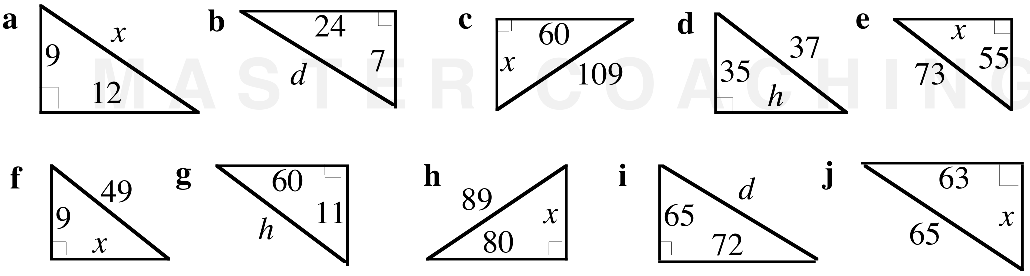
Fly of course could have *flown* 'the body diagonal' FS to reach Spider,
the airial distance being $FS = \sqrt{4 \cdot 8^2 + 3 \cdot 4^2 + 2 \cdot 5^2} = 6 \cdot 39$ to 2 dec places

Master Coaching

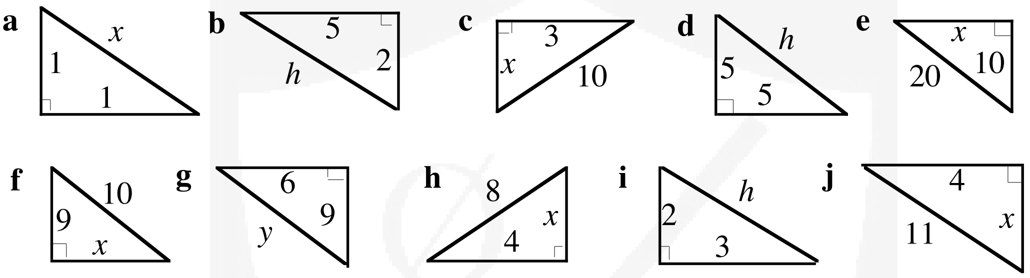
Exercises : **Pythagoras' Theorem**

Note that these diagrams are <u>not</u> drawn to scale.

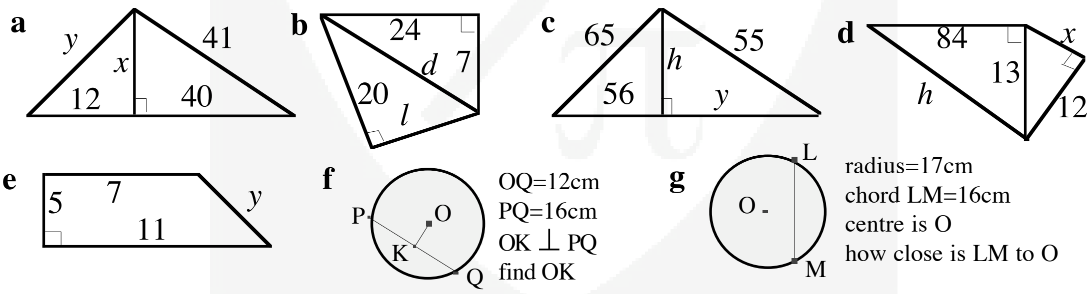
1 Find the value of the pronumerals in these diagrams. All these answers are *integers*.



2 Find the value of the pronumerals in these diagrams. All these answers are *surds*.



3 Find the value of the pronumerals in these diagrams.



4 **a** What is the longest thin needle that can fit inside a box 12cm by 6cm by 4cm?
b Clarisse hikes 8km east and then 5km south. How far is she from her starting place?
c A rhombus of side17cm has a long diagonal of 30cm. How long is the short diagonal?
d A bee goes 9m west, then 4m north, then up 3m. How far is it from its starting place?

Answers ... you can simplify some of these surds ...

1 **a** 15 **b** 25 **c** 91 **d** 12 **e** 48 **f** 40 **g** 61 **h** 39 **i** 97 **j** 16

2 **a** $\sqrt{2}$ **b** $\sqrt{29}$ **c** $\sqrt{91}$ **d** $\sqrt{50}$ **e** $\sqrt{300}$ **f** $\sqrt{19}$ **g** $\sqrt{117}$ **h** $\sqrt{48}$ **i** $\sqrt{13}$ **j** $\sqrt{105}$

3 **a** 9,15 **b** 25,15 **c** 33,44 **d** 85,5 **e** $\sqrt{41}$ **f** $\sqrt{80}$ cm **g** 15cm

4 **a** 14cm **b** $\sqrt{89}$ km **c** 16cm **d** $\sqrt{106}$ m

Surds : see separate set of notes in this series ... " Master Coaching Surds "

Multiplying $\times$ and dividing $\div$ are strong operations and can 'break' the square root sign :

$\sqrt{\mathbf{A}\times\mathbf{B}}=\sqrt{\mathbf{A}}\times\sqrt{\mathbf{B}}$ $\quad\sqrt{21}=\sqrt{3\times7}=\sqrt{3}\times\sqrt{7}$ $\quad\sqrt{48}=\sqrt{6\times8}=\sqrt{2}\times\sqrt{24}=\sqrt{3}\times\sqrt{16}$

$\sqrt{\mathbf{A}}\times\sqrt{\mathbf{B}}=\sqrt{\mathbf{A}\times\mathbf{B}}$ $\quad\sqrt{4}\times\sqrt{5}=\sqrt{20}$ $\quad\sqrt{7}\times\sqrt{8}=\sqrt{56}$ $\quad\sqrt{5}\times\sqrt{5}=\sqrt{25}$

$\sqrt{\mathbf{A}\div\mathbf{B}}=\sqrt{\mathbf{A}}\div\sqrt{\mathbf{B}}$ $\quad\sqrt{34\div2}=\sqrt{34}\div\sqrt{2}$ $\quad\sqrt{18\div3}=\sqrt{18}\div\sqrt{3}$

$\sqrt{\frac{\mathbf{A}}{\mathbf{B}}}=\frac{\sqrt{\mathbf{A}}}{\sqrt{\mathbf{B}}}$ $\quad\sqrt{\frac{28}{7}}=\frac{\sqrt{28}}{\sqrt{7}}$ $\quad\sqrt{\frac{55}{11}}=\frac{\sqrt{55}}{\sqrt{11}}$ $\quad\sqrt{\frac{91}{13}}=\frac{\sqrt{91}}{\sqrt{13}}$

Addition $+$ and Subtraction $-$ are weak, and <u>can not</u> 'break' the square root sign :

$\sqrt{A+B}\neq\sqrt{A}+\sqrt{B}$ $\quad\sqrt{16+9}\neq\sqrt{16}+\sqrt{9}$ $\quad\sqrt{7^2+3^2}\neq7+3$

$\sqrt{A-B}\neq\sqrt{A}-\sqrt{B}$ $\quad\sqrt{16-9}\neq4-3$ $\quad\sqrt{6^2-2^2}\neq6-2$

By definition $\sqrt{\mathbf{A}^2}=\mathbf{A}$, and of course $\left(\sqrt{\mathbf{A}}\right)^2=\mathbf{A}$ ie $\sqrt{\mathbf{A}}\times\sqrt{\mathbf{A}}=\mathbf{A}$

$\sqrt{7^2}=7$, and of course $\left(\sqrt{7}\right)^2=7$ ie $\sqrt{7}\times\sqrt{7}=\sqrt{49}=7$

Factorising ... keep the same power :

$10^2=2^2\times5^2$ $\quad10^3=2^3\times5^3$ $\quad12^7=3^7\times4^7$ $\quad15^{11}=3^{11}\times5^{11}$ $\quad28^5=4^5\times7^5$

$21^2+35^2=7^2.(3^2+5^2)$ $\quad8^2+6^2=2^2.(4^2+3^2)$ $\quad22^2+33^2=11^2.(2^2+3^2)$

$15^2+12^2=3^2.(5^2+4^2)$ $\quad14^8+21^8=7^8.(2^8+3^8)$ $\quad27^2+63^2=9^2.(3^2+7^2)$

In Pythagoras questions (and solving quadratic equations) the power will only be 2.

Pythagoras ... calculations can sometimes be made simpler by the following logic :

$h^2=6^2+10^2$	$d^2=25^2+20^2$	$l^2=28^2-21^2$	$x^2=77^2-66^2$
$h^2=2^2\times(3^2+5^2)$	$d^2=5^2\times(5^2+4^2)$	$l^2=7^2\times(4^2-3^2)$	$x^2=11^2\times(7^2-6^2)$
$h=\sqrt{2^2\times(3^2+5^2)}$	$d=\sqrt{5^2\times(5^2+4^2)}$	$l=\sqrt{7^2\times(4^2-3^2)}$	$x=\sqrt{11^2\times(7^2-6^2)}$
$h=\sqrt{2^2}\times\sqrt{3^2+5^2}$	$d=\sqrt{5^2}\times\sqrt{5^2+4^2}$	$l=\sqrt{7^2}\times\sqrt{4^2-3^2}$	$x=\sqrt{11^2}\times\sqrt{7^2-6^2}$
$h=2\times\sqrt{9+25}$	$d=5\times\sqrt{25+16}$	$l=7\times\sqrt{16-9}$	$x=11\times\sqrt{49-36}$
$h=2\sqrt{34}$	$d=5\sqrt{41}$	$l=7\sqrt{7}$	$x=11\sqrt{13}$

Of course with a bit of practice you can do *all* of this in your head!
Can you visualise the steps in these :

$h^2=12^2+18^2$	$d^2=17^2+34^2$	$l^2=70^2-40^2$	$x^2=54^2-45^2$
$h=$	$d=$	$l=$	$x=$
$h=6\sqrt{13}$	$d=17\sqrt{5}$	$l=10\sqrt{33}$	$x=9\sqrt{11}$