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# Year 12 2 Unit Extension 1

# Test #9 Answers

## Question 1

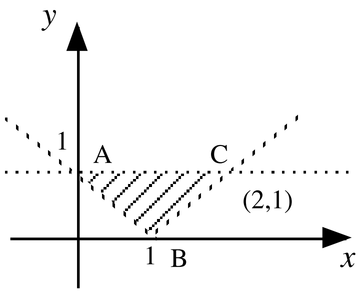
**a i** $f(x) = \ln\left(x^2 + 2x\right)$

$$f'(x) = \frac{2(x+1)}{x^2 + 2x}$$

**ii** $f(x) = e^{x^2}.\cos\left(x^2\right)$

$$f'(x) = 2xe^{x^2}\cos\left(x^2\right) - 2xe^{x^2}\sin\left(x^2\right)$$

**b**



Interior, but not the boundary of triangle joining A(0,1), B(1,0), C(2,1)

**c i** $$\int_0^2 xe^{x^2}.dx = \frac{1}{2}e^{x^2}\Big|_0^2$$

$$= 26\cdot 7991$$

**ii** $$\int_0^1 \frac{x+2}{\sqrt{1-x^2}}.dx = -\sqrt{1-x^2} + 2\sin^{-1}(x)\Big|_0^1$$

$$= 1 + \pi$$

Note : $$\int_0^1 \frac{x+2}{\sqrt{1-x^2}}dx = \underset{\text{Use } u=1-x^2}{\underset{\nearrow}{\int_0^1 \frac{xdx}{\sqrt{1-x^2}}}} + \underset{\text{Standard Integral}}{\underset{\nearrow}{\int_0^1 \frac{2dx}{\sqrt{1-x^2}}}}$$

**d i** Pr (2H’s given 1 is H) $= \frac{1}{3}$

**ii** Pr (2H's given $H_{10\,c}$) $= \frac{1}{2}$

## Question 2

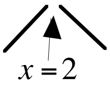
**a** Area of Rectangle as $2xy$ where $y = \frac{4}{x^2+4}$ $\therefore$ $A(x) = \frac{8x}{x^2+4}$

Area is a MAX when $A'(x) = 0$ and $A''(x) < 0$

$$A(x) = \frac{8\left(x^2+4\right) - 16x^2}{\left(x^2+4\right)^2} \quad \therefore \quad A'(x) = 0 \text{ when } 32 - 8x^2 = 0$$

$$x = \pm 2$$

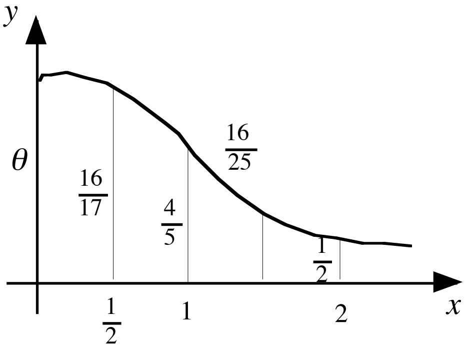
$\therefore \quad A'(2^+)<0$ and $A'(2^-)<0$



$\Rightarrow \quad A(2)$ is a MAX value

$\therefore \quad$ MAX Area $= \frac{16}{8} = 2$ units squared

**b** $A_{\max} = 2\int_0^2 \frac{4}{x^2+4}.dx$



By Simpson's Rule $A_{\max} \approx 2\times\frac{1}{6}\left[1+4\times\frac{16}{17}+2\times\frac{4}{5}+4\times\frac{16}{25}+\frac{1}{2}\right]$

$\approx \frac{1}{3}\left[9\frac{361}{850}\right]$

$\approx 3{\cdot}141568$ or $3{\cdot}1416\,(4\,\text{dp})$

**c** Exact value $= 2\int_0^2 \frac{4}{x^2+4}dx = 4\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^2$

$= \pi . \approx 3{\cdot}1416\,(4\,\text{dp})$

$\therefore \quad$ S.R. gives the exact answer of $\pi$ to 4dp

## Question 3

**a** $P(1)$, $n=1$, LHS = 5 RHS $= \frac{5+3\times25}{16}$ $\therefore$ $P(1)$ is true

$=5$

$P(k)$ Assume true for $n=k$ ie Assume $1\times5+2\times5^2+...+k\times5^k = \frac{5+(4k-1)\times5^{k+1}}{16}$

$P(k+1)$ Prove, using $P(k)$ that formula is true for $n=k+1$

ie Prove $1\times 5+2\times 5^2+\ldots+k\times 5^k+(k+1)5^{k+1}=\dfrac{5+(4k+3)\times 5^{k+2}}{16}$

$$\begin{aligned}\text{LHS} &= 1\times 5+2\times 5^2+\ldots+k\times 5^k+(k+1)5^{k+1}\\ &= \frac{5+(4k-1)\times 5^{k+1}}{16}+(k+1)5^{k+1}\\ &= \frac{5+(4k-1)5^{k+1}+(16k+16)5^{k+1}}{16}\\ &= \frac{5+(20k+15)\times 5^{k+1}}{16}\\ &= \frac{5+(4k+3)\times 5\times 5^{k+1}}{16}\\ &= \frac{5+(4k+3)\times 5^{k+2}}{16}\end{aligned}$$

Q.E.D

**b** $f(x)=x-\ln(x+1) \qquad 0<x<1$

$f'(x)=1-\dfrac{1}{x+1}=\dfrac{x}{x+1}$

Now $f''(x)=\dfrac{1}{(x+1)^2}\neq 0$ for any $x$

$\therefore$ $f'(x)$ has no turning points and its MAX and MIN values are at the extremities

$\therefore$ min $f'(x)=f'(0)=0$

max $f'(x)=f'(1)=\dfrac{1}{2}$

for $x\in(0,1)$, $\dfrac{x}{2}<\dfrac{x}{x+1}$ because $x+1<2$

and

$\dfrac{x}{x+1}>\dfrac{x}{1}$ because $x+1>1$

$\therefore$ $\dfrac{x}{2}<f'(x)<x$

$$\therefore \int_0^x \frac{x}{2}\,dx<\int_0^x f'(x)\,dx<\int_0^x x\,dx$$

ie $\frac{x^2}{4} < x - \ln(x+1) < \frac{x^2}{2}$ × 12 and integrate

$$\int_0^1 3x^2 dx < \int_0^1 (12x - 12\ln(x+1))dx < \int_0^1 6x^2 dx$$

$$1 < 6 - \int_0^1 \ln(x+1)^{12} dx < 2$$ multiply by $-1$ and change

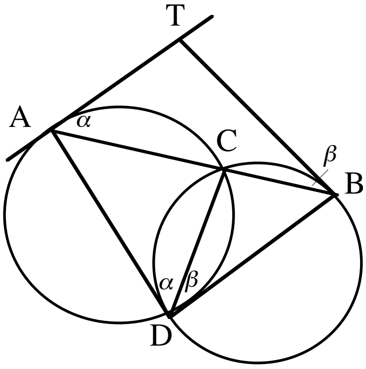
$$\therefore \quad -2 < \int_0^1 \ln(x+1)^{12} dx - 6 < -1$$ add 6

$$4 < \int_0^1 \ln(x+1)^{12} dx < 5$$

## Question 4

**a** "The angle between a tangent to a circle and a chord drawn to the point of contact......"
- is equal to any angle in the alternate segment

**.b i**



Join AD, DB & DC

Let $\angle TBA = \beta$ and $\angle TAB = \alpha$ as shown

$\therefore \ \angle ATB = 180° - (\alpha + \beta)$
(sum of angles of ΔTAB)

Proof $\angle ADC = \angle TAB = \alpha$ (By Theorem part **1a**)
$\angle BDC = \angle TBA = \beta$ (By Theorem part **1a**)
$\therefore$ $\angle ADB = \alpha + \beta$
$\therefore$ $\angle ADB$ and $\angle ATB$ are supplementary angles
(Shown $\angle ADB = \alpha + \beta$, $\angle ATB = 180° - (\alpha + \beta)$)

$\therefore$ TADB is a cyclic quadrilateral Q.E.D.

**ii** $x = 65°$ (opp ∠'s of cyclic quad)
$y = 103°$ (ext ∠ of cyclic quad)

**c i** By Division Transformation

$x^{10}+7x^2-3x+5\equiv\left(x^2-1\right)\mathrm{Q}(x)+ax+b$

where $ax+b$ is the remainder on division by $\left(x^2-1\right)$

$x=1\ \Rightarrow\ 10=a+b \qquad \therefore\ \ a=-3$

$x=-1\ \Rightarrow\ 16=-a+b \qquad b=13$

$\therefore$ Remainder is $-3x+13$

**c ii** Let the roots of equation $4x^3-8x^2+6x-7=0$ be $x$.

We want the equation with roots $y=\dfrac{1}{x}$; ie $x=\dfrac{1}{y}$

$\therefore$ Required equation is $\dfrac{4}{y^3}-\dfrac{8}{y^2}+\dfrac{6}{y}-7=0$ or $7y^3-6y^2+8y-4=0$

$\therefore\ \sum y=\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{6}{7}$

Alternate Method :

$4x^3-8x^2+6x-7=0$ has roots $a$, $b$, $c$

$\therefore\ \alpha+\beta+\gamma=2 \qquad \alpha\beta+\alpha\gamma+\beta\gamma=\dfrac{6}{4}$

$\alpha\beta\gamma=\dfrac{7}{4}$

$$\therefore\ \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\lambda}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$$

$$=\frac{6}{4}\div\frac{7}{4}$$

$$=\frac{6}{7}$$

## Question 5

**a** $\left(3x-\dfrac{2}{x^2}\right)^{15} \qquad \mathrm{T}_{k+1}={}^{15}\mathrm{C}_k\,(3x)^{15-k}\left(-\dfrac{2}{x^2}\right)^k$

Indep of $x\Rightarrow 15-3k=0 \qquad \therefore\ \mathrm{T}_6={}^{15}\mathrm{C}_5\,3^{10}\times(-2)^5$

$k=5 \qquad =-5674372704$

**b** Soln $x=2\cos nt \qquad \therefore\ n^2=\dfrac{9}{4},\ n=\dfrac{3}{2}$

$\dot{x}=-2n\sin nt$

$\ddot{x}=-2n^2\cos nt \qquad \therefore\ x=2\cos\dfrac{3}{2}t$

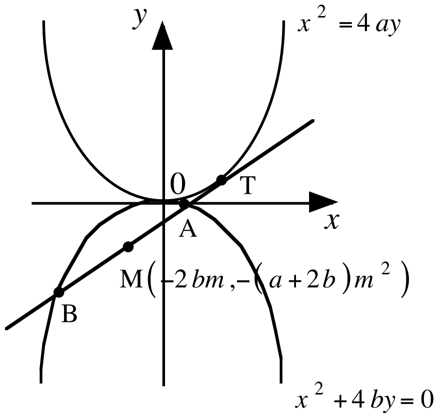
$=-n^2x$

**i** $$\frac{dx}{dt} = \pm 3\sqrt{1-\cos^2 nt}$$
$$= \pm 3\sqrt{1-\frac{x^2}{4}}$$
$$= \pm \frac{3}{2}\sqrt{4-x^2}$$

**ii** $$\frac{dx}{dt} = -2 \times \frac{3}{2}\sin\frac{3}{2}t$$
$$= -3\sin nt$$

$$n = \frac{3}{2}$$

## Question 6



Line $y = mx + c$ is a tangent to $x^2 = 4ay$
$\Leftrightarrow$ $x^2 = 4a(mx + x)$ has a double root
ie $x^2 - 4amx - 4ac = 0$ has a double root
ie $\Delta = 16a^2m^2 + 16ac = 0$
$c = -am^2$

Now
$y = mx + c$ cuts $x^2 + 4by = 0$ at A, B as shown

when
$x^2 + 4b(mx + c) = 0$ $\quad\quad c = -am^2$
$x^2 + 4bmx + 4bc = 0$

$$x = \frac{-4bm \pm \sqrt{\phantom{x}}}{2}$$

$\therefore$ Mid -point of interval AB is $x = -\frac{4bm}{2}$
$x = -2bm$

$y = m(-2bm) - am^2$ $\quad\quad m = \frac{x}{-2b}$
$= -(a + 2b)m^2$

subst for $x$ to obtain locus of mid-pt

$\therefore$ $y = -(a+2b).\frac{x^2}{4b^2}$

ie $x^2 = -4\left(\dfrac{b^2}{a+2b}\right)y$ which is a parabola with focal length $= \dfrac{b^2}{a+2b}$

## Question 7

**a** By Newton's Law of Cooling

$$\frac{d\mathrm{T}}{dt} = -k(\mathrm{T}-20)+5$$

$$\therefore \quad \frac{d\mathrm{T}}{dt} = -k\left(\mathrm{T}-\left(20+\frac{5}{k}\right)\right) \qquad \text{Let } 20+\frac{5}{k} = A$$

$$\therefore \quad \frac{d\mathrm{T}}{dt} = -k(\mathrm{T}-\mathrm{A})$$

$$\therefore \quad \int \frac{d\mathrm{T}}{\mathrm{T}-\mathrm{A}} = \int -k\,dt$$

$$\ln(\mathrm{T}-\mathrm{A}) = -kt+\mathrm{C}$$

$$\mathrm{T}-\mathrm{A} = \mathrm{T}_0 e^{-kt} \qquad \left(\mathrm{T}_0 = e^{\mathrm{C}}\right)$$

$$\therefore \quad \mathrm{T} = \mathrm{A}+\mathrm{T}_0 e^{-kt}$$

Now $\lim\limits_{t\to\infty} \quad \mathrm{T} = \mathrm{A} = 120 = 20+\dfrac{5}{k}$

$$\therefore \ \frac{5}{k} = 100,\ k = 0\cdot05$$

$$\therefore \quad \mathrm{T} = 120+\mathrm{T}_0 e^{-0\cdot05\,t}$$

$$t=0,\ \mathrm{T}=20 \Rightarrow 20 = 120+\mathrm{T}_0$$

$$\therefore \ \mathrm{T}_0 = -100$$

$$\therefore \ \mathrm{T} = 120-100e^{-0\cdot05\,t}$$

$$\therefore \quad \mathrm{T} = 100 \ \text{ when}$$

$$10\cancel{0} = 12\cancel{0} - 10\cancel{0}e^{-0\cdot05\,t}$$

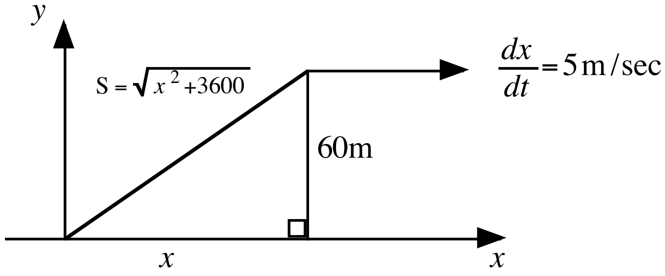
$$\text{ie} \quad 10e^{-0\cdot05\,t} = 2$$

$$e^{0\cdot05t} = 5$$

$$t = \frac{\ln5}{0\cdot05}$$

$$\approx 32\cdot2\,\text{min}$$

**b**



Length of String $\quad S=\sqrt{x^2+3600}$

$$\therefore \quad \frac{dS}{dx}=\frac{x}{\sqrt{x^2+3600}}$$

$$\therefore \quad \frac{dS}{dt}=\frac{dS}{dx}\times\frac{dx}{dt} \qquad \text{when } S=100$$

$$=\frac{x}{\sqrt{x^2+3600}}\times 5$$

Note: when
$S=100,\ x=80$

$$=\frac{80}{100}\times 5\,\text{m/s}$$

$$=4\,\text{m/s}$$

$\therefore$ Boy must pay out string at 4m/s when kite 100m away from him.